Topological Dynamics and C^*-Algebras

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Abstract

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If G is a group of automorphisms of a C^*-algebra A with identity, then G acts in a natural way as a transformation group on the state space $S(A)$ of A. Moreover, this action is uniformly almost periodic iff G has compact pointwise closure in the space of all maps of A into A. Consideration of the enveloping semigroup of $(S(A),G)$ shows that in this case, this pointwise closure $\bar{G}$ is a compact topological group consisting of automorphisms of A. The Haar measure on $\bar{G}$ is used to define an analogue of the canonical center-valued trace in a finite von Neumann algebra. If A possesses a sufficiently large group G_o of inner automorphisms such that $(S(A),G_o)$ is uniformly almost periodic, then A is a central C^*-algebra. The notion of a uniquely ergodic system is applied to give necessary and sufficient conditions that an approximately finite C^*-algebra possess exactly one finite trace.

Introduction

The purpose of this paper is to apply some ideas from topological dynamics to the study of C^*-algebras. If X is a compact Hausdorff space and (X,Γ) is a topological transformation group, then Γ has a natural representation as a group of automorphisms of the commutative C^*-algebra $C(X)$: for $t \in \Gamma$ and $f \in C(X)$ put

$$(tf)(x) = f(xt) , \qquad\qquad x \in X .$$

It is often possible to express properties of (X,Γ) in terms of the system $(\Gamma,C(X))$; for example, (X,Γ) is uniformly almost periodic iff for each $f \in C(X)$, the set $\{tf: t \in \Gamma\}$ is relatively compact in $C(X)$. If A is an arbitrary C^*-algebra with identity and G is a group of automorphisms of A , we may view the pair (G,A) as a non-commutative version of $(\Gamma,C(X))$. We shall see that some of the relationships between (X,Γ) and $(\Gamma,C(X))$ have non-commutative analogues, and that these analogues can be used to obtain information about the structure of certain C^*-algebras.

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1. Preliminaries.

We shall generally follow the terminology of [6] for topological dynamics, that of [5] for C^*-algebras, and that of [12] for uniform spaces and topologies on function spaces. We shall however translate Dixmier's "morphisme" by "*-homomorphism", and we define "trace" below.

<u>Definition</u> 1.1. Let G be a semigroup with identity e, and let X be a set. A <u>right action</u> of G on X is a mapping

$$\pi : X \times G \to X : (x,\alpha) \to x\alpha = \pi(x,\alpha)$$

such that

1) $xe = x$ for all $x \in X$, and

2) $(x\alpha)\beta = x(\alpha\beta)$ for all $x \in X$ and all $\alpha, \beta \in G$.

When there is no danger of confusion, we shall write $x\alpha$ for $\pi(x,\alpha)$.

A <u>left action</u> of G on X is a mapping $(\alpha, x) \to \alpha x$ of $G \times X$ into X such that $ex = x$ and $\alpha(\beta x) = (\alpha\beta)x$ for all $x \in X$ and all $\alpha, \beta \in G$. We make the convention that the term "action" will mean "right action" unless we specify otherwise. An action is <u>continuous</u> if it is continuous from the product topology on $X \times G$.

<u>Remark</u> 1.2. If X and Y are sets, we shall write Y^X for the set of all mappings of X into Y. When $Y = X$, we may use composition of mappings to provide X^X with two natural semigroup structures : Let (p,q) be an ordered pair of elements of X^X. If we write our mappings on the right, we shall define

pq by

$$x(pq) = (xp)q \, , \qquad x \in X \, .$$

If we write our mappings on the left, we shall compose in the opposite order:

$$(pq)x = p(qx) \, , \qquad x \in X \, .$$

These definitions give actions of X^X on X on the right and left respectively. We shall find it convient to write maps of C^*-algebras on the left and maps of their state spaces on the right.

Remark 1.3. Let A be a C^*-algebra with identity. We write $S(A)$ for its state space, and we give $S(A)$ the weak* topology. Then the set $S(A)$ is convex, and the topological space $S(A)$ is compact and Hausdorff, hence is a uniform space in a unique way. We note that the uniformity on $S(A)$ is determined by the family of all pseudo-norms of the form

$$(p,q) \to |p(a) - q(a)| \, ,$$

where a is a positive element of A. It follows that a net ρ_γ in $S(A)^{S(A)}$ converges to $\rho \in S(A)^{S(A)}$ in the topology of uniform convergence iff for each positive $a \in A$ we have

$$\sup_{p \in S(A)} |(p\rho_\gamma)(a) - (p\rho)(a)| \to 0 \quad [12, \text{p.}226\text{–}227].$$

We note also that as $S(A)$ is compact, a family of maps in $S(A)^{S(A)}$ is equicontinuous iff it is uniformly equicontinuous.

We write $ES(A)$ for the set of pure states of A with the weak* topology.

<u>Remark</u> 1.4. Let (X,G) be a transformation group with compact Hausdorff phase space X. For each $t \in G$, let π^t denote the map $x \to xt$, $x \in X$. The pointwise closure in X^X of the set $\{\pi^t : t \in G\}$ is a semigroup, called the enveloping semigroup of (X,G) [6, 3.2]. The following are equivalent:

1) (X,G) is uniformly almost periodic;

2) $\{\pi^t : t \in G\}$ is an equicontinuous family;

3) The enveloping semigroup of (X,G) is a group of continuous maps;

4) If $f \in C(X)$, then f is almost periodic, i.e. $\{f \circ \pi^t : t \in G\}$ has compact closure in $C(X)$ [6, 4.4 and 4.15].

(The proof given in [6, 4.15] for real functions applies equally well to $C(X)$.)

Let A be a C^*-algebra with identity. If $p \in S(A)$, we write L^p for the representation of A obtained by applying the Gelfand-Naimark-Segal construction to p, and we say that L^p is <u>associated</u> to p. The <u>left kernel</u> of p is the left ideal $\{a \in A : p(a^*a) = 0\}$. A state τ of A is a <u>trace</u> on A if τ is invariant under the inner automorphisms of A, i.e. $\tau(a) = \tau(uau^*)$ for all $a \in A$ and all unitary $u \in A$. Since every element of A is a linear combination of unitaries, a state τ of A is a trace iff $\tau(ab) = \tau(ba)$ for all $a,b \in A$. We denote the set of all traces on A by $T(A)$, and we write $ET(A)$ for the set of extremal traces of A, i.e. extreme points of $T(A)$. A trace τ is extremal iff L^τ is a factor representation [5, 6.7.3 and 6.8.5].

A <u>face</u> of a compact convex set K is a convex subset F of K such that if $p,q \in K$ and $\frac{1}{2}p + \frac{1}{2}q \in F$, then p and q

are in F. An extreme point of a face of K is also an extreme point of K, and the inverse image of an extreme point under an affine map is a face.

If A is a C^*-algebra, we denote by $\operatorname{Max}(A)$ the space of all maximal ideals of A equipped with the relative topology from the Jacobson topology on $\operatorname{Prim}(A)$ [5, 3.1.1]. We write ZA for the center of A. Suppose A has an identity. Then there is a mapping ρ of $\operatorname{Prim}(A)$ onto $\operatorname{Max}(ZA)$ given by $\rho : P \to P \cap ZA$. This mapping is continuous, and since $\operatorname{Prim}(A)$ is compact and $\operatorname{Max}(ZA)$ is Hausdorff, it is also closed. If ρ is one-to-one (i.e. a homeomorphism), then A is said to be a <u>central</u> C^*-algebra [1].

<u>Remark</u> 1.5. Let A be a C^*-algebra with identity I, and let p be a state of A such that L^p is a factor representation (e.g. a pure state or an extremal trace). We identify the center of $L^p(A)'$ with $\mathbb{C}$. Then p coincides with L^p on ZA, hence is multiplicative on ZA. Suppose moreover that $\ker L^p$ is a primitive ideal. Then the character of ZA which corresponds to $\ker L^p \cap ZA$ is $p|_{ZA}$. For if the value of this character on z is λ, then $z - \lambda I \in \ker L^p \cap ZA \subseteq \ker L^p$, so $p(z-\lambda I) = L^p(z-\lambda I) = 0$, and $p(z) = \lambda$.

2. Uniformly Almost Periodic Groups of Automorphisms.

In this section A will denote a C^*-algebra with identity I. An <u>automorphism</u> of A is an invertible $*$-homomorphism of A onto A, and we write $\mathrm{Aut}(A)$ for the group of all automorphisms of A. We shall characterize those subgroups of $\mathrm{Aut}(A)$ which act uniformly almost periodically on $S(A)$.

Let G be a subgroup of $\mathrm{Aut}(A)$. We say that an element a of A is G-<u>invariant</u> if $\alpha(a) = a$ for all $\alpha \in G$. A state p of A is G-invariant if $p \circ \alpha = p$ for all $\alpha \in G$. We denote the algebra of all G-invariant elements of A by $Z_G A$ and the set of all G-invariant states of A by $S_G(A)$. Then $Z_G A$ is a C^*-subalgebra of A, and $S_G(A)$ is a compact convex subspace of $S(A)$.

Let A^A have the pointwise (product) topology. Then the set $\mathrm{Aut}(A)$ is not in general closed in A^A, since a net of automorphisms may converge pointwise to a map which is not onto. It will therefore be convenient for us to consider a slightly larger subset of A: Let $H(A)$ be the set of all $*$-homomorphisms α of A into A such that $\alpha(I) = I$. Then $H(A)$ is pointwise closed, the elements of $H(A)$ are norm-decreasing positive maps, and an element of $H(A)$ is an automorphism iff it is an invertible mapping. Moreover, $H(A)$ is closed under composition of mappings, hence is a subsemigroup of A^A. We note that a net $\{\alpha_\gamma\}$ converges to α in $H(A)$ iff $\alpha_\gamma(a) \to \alpha(a)$ for each positive $a \in A$.

<u>Lemma</u> 2.1. $H(A)$ is a topological semigroup, and $\mathrm{Aut}(A)$ is a topological group.

<u>Proof</u>: Suppose $(\alpha_\gamma, \beta_\gamma) \to (\alpha, \beta)$ in $H(A) \times H(A)$, and let $a \in A$.

Then

$$\|\alpha_\gamma\beta_\gamma(a) - \alpha\beta(a)\| \leq \|\alpha_\gamma\| \;\|\beta_\gamma(a) - \beta(a)\| + \|\alpha_\gamma(\beta(a)) - \alpha(\beta(a))\| \;.$$

As $\alpha_\gamma \to \alpha$, $\beta_\gamma \to \beta$, and $\|\alpha_\gamma\| \leq 1$ for all γ , this tends to zero, so $\alpha_\gamma\beta_\gamma \to \alpha\beta$. Thus $H(A)$ is a topological semigroup.

To show that $\mathrm{Aut}(A)$ is a topological group we suppose that $\alpha_\gamma \to \alpha$ in $\mathrm{Aut}(A)$. Let $a \in A$. Automorphisms of C^*-algebras are isometric, so

$$\|\alpha_\gamma^{-1}(a) - \alpha^{-1}(a)\| = \|a - \alpha_\gamma\alpha^{-1}(a)\| = \|\alpha(\alpha^{-1}(a)) - \alpha_\gamma(\alpha^{-1}(a))\| \to 0 \;.$$

Thus inversion is continuous on $\mathrm{Aut}(A)$, and $\mathrm{Aut}(A)$ is a topological group.

If $\alpha \in H(A)$ and p is a state of A , then $p \circ \alpha$ is again a state of A . Thus there is a natural action of $H(A)$ on $S(A)$ defined by

$$(p,\alpha) \to p \circ \alpha = p\alpha \,, \qquad p \in S(A) \,, \quad \alpha \in H(A) \;.$$

This action is continuous : if $(p_\gamma,\alpha_\gamma) \to (p,\alpha)$ in $S(A) \times H(A)$ and $a \in A$, then

$$|p_\gamma \circ \alpha_\gamma(a) - p \circ \alpha(a)| \leq \|\alpha_\gamma(a) - \alpha(a)\| + |p_\gamma(\alpha(a)) - p(\alpha(a))| \to 0 \;.$$

It follows that if G is any subgroup of $\mathrm{Aut}(A)$, then the restriction of this action to $S(A) \times G$ makes $(S(A),G)$ into a transformation group.

Since $H(A)$ is closed in A^A , we have for any subset G of $H(A)$ that the closures of G in $H(A)$ and in A^A coincide. We shall find the following theorem very useful in providing examples of uniformly almost periodic actions on state spaces.

<u>Theorem</u> 2.2. Let G be a subset of $H(A)$, and let S be any subset of A such that the linear span of S is dense in A.

Then the closure of G in $H(A)$ is compact iff for every $a \in S$, the set $G[a] = \{\alpha(a) : \alpha \in G\}$ has compact closure in A.

<u>Proof</u>: Let $\bar{G}$ be the closure of G. As $\alpha \to \alpha(a)$ is continuous, we have $\bar{G}[a] = \{\alpha(a) : \alpha \in \bar{G}\} \subseteq \overline{G[a]}$ for every $a \in A$.

If $\bar{G}$ is compact, then for each $a \in A$ we have $\bar{G}[a]$ compact, whence $\bar{G}[a] = \overline{G[a]}$. In particular, $\overline{G[a]}$ is then compact for every $a \in S$.

Conversely, suppose that for every $a \in S$, the set $\overline{G[a]}$ is compact. The elements of $\bar{G}$ are linear and norm-decreasing, so the restriction mapping $r : \alpha \to \alpha|_S$ is a one-to-one map of $\bar{G}$ into A^S. Let A^S have the product topology. Then $\alpha_\gamma \to \alpha$ in $\bar{G}$ iff $r(\alpha_\gamma) \to r(\alpha)$ in A^S. For if $r(\alpha_\gamma) \to r(\alpha)$, then $\alpha_\gamma \to \alpha$ pointwise on S, hence pointwise on the linear span of S. As this span is dense, and as $\bar{G}$ consists of maps uniformly bounded in norm, an $\epsilon/3$-argument shows that $\alpha_\gamma(a) \to \alpha(a)$ for all $a \in A$.

Thus r is a homeomorphism of $\bar{G}$ onto its image in A^S. To see that this image is closed in A^S, suppose $r(\alpha_\gamma)$ is a net in the image such that $r(\alpha_\gamma) \to \theta$ in A^S. It is enough to show that θ has an extension to a map α in A^A such that $\alpha_\gamma \to \alpha$ in A^A. We extend θ to the linear span of S by $\theta(\sum_{i=1}^{n} \lambda_i a_i) = \sum_{i=1}^{n} \lambda_i \theta(a_i)$. We have

$$\sum_{i=1}^{n} \lambda_i \theta(a_i) = \sum_{i=1}^{n} \lambda_i \lim_\gamma \alpha_\gamma(a_i) = \lim_\gamma \alpha_\gamma(\sum_{i=1}^{n} \lambda_i a_i) .$$

It follows that θ is well-defined and linear and that α_γ converges pointwise to θ on the span of S. Since the α_γ are norm-decreasing, θ is also norm-decreasing on this span. Thus θ has an extension to a linear, norm-decreasing map α of A into A. By the same $\epsilon/3$-argument as above, $\alpha_\gamma \to \alpha$ in A^A.

To complete the proof we observe that the image of $\overline{G}$ is contained in $\times_{a\in S} \overline{G}[a] \subseteq \times_{a\in S} \overline{G[a]}$ and apply the Tychonoff Theorem.

We shall need to consider more closely the maps of $S(A)$ into itself which are induced by the elements of $H(A)$. For each $\alpha \in H(A)$, let $i(\alpha)$ be the mapping $p \to p\alpha$. Then i is an injection of $H(A)$ into $S(A)^{S(A)}$, for if $p(\alpha(a)) = p(\beta(a))$ for all $p \in S(A)$ and all $a \in A$, then $\alpha(a) = \beta(a)$ for all $a \in A$. Moreover, i has the following additional properties:

1) i takes the identity of the semigroup $H(A)$ onto that of the semigroup $S(A)^{S(A)}$, and i is a homomorphism of semigroups: $i(\alpha\beta) = i(\alpha)i(\beta)$.

2) For each $\alpha \in H(A)$, $i(\alpha)$ is weak*-continuous and affine.

3) If $\alpha \in H(A)$ and α is invertible, then $i(\alpha)$ is invertible and $i(\alpha)^{-1} = i(\alpha^{-1})$.

<u>Lemma</u> 2.3. The map i is bicontinuous from $H(A)$ into $S(A)^{S(A)}$ when $S(A)^{S(A)}$ is given the topology of uniform convergence.

<u>Proof</u>: If $a \in A$ is positive and $\alpha_\gamma, \alpha \in H(A)$, then $\alpha_\gamma(a) - \alpha(a)$ is self-adjoint. Hence $\|\alpha_\gamma(a) - \alpha(a)\| = \sup_{p\in S(A)} |p(\alpha_\gamma(a) - \alpha(a))|$.

<u>Lemma</u> 2.4. Let K be a convex subset of the dual of a Banach space B, and let K have the weak* topology. Let F be a family of affine maps of K into K. Then the pointwise closure of F in K^K is again a family of affine maps.

<u>Proof</u>: Let $\lambda \in [0,1]$, let $p,q \in K$, and suppose β_γ converges pointwise in K^K. Then the functionals $\lim_\gamma \beta_\gamma(\lambda p + (1-\lambda)q)$ and

$\lambda \lim_\gamma \beta_\gamma(p) + (1-\lambda) \lim_\gamma \beta_\gamma(q)$ agree on each element of B.

Theorem 2.5. Let G be a subgroup of $\mathrm{Aut}(A)$. Then the following are equivalent:

1) The transformation group $(S(A),G)$ is uniformly almost periodic;

2) The closure $\overline{G}$ of G in $H(A)$ (or in A^A) is compact.

Under these conditions $\overline{G}$ is a group, and i is a homeomorphism and a group isomorphism of $\overline{G}$ onto the enveloping semigroup E of $(S(A),G)$.

Proof: Let T be the topology of uniform convergence. We use below without commont Remark 1.4 and some topological results which can be found in [12, pp. 232-233 and p. 227].

Suppose $\overline{G}$ is compact. Then $i(\overline{G})$ is T-compact by Lemma 2.3. The topology T is jointly continuous on the family of all continuous maps of $S(A)$ into $S(A)$, so $i(\overline{G})$ is equicontinuous. In particular, the subfamily $i(G)$ is equicontinuous, so $(S(A),G)$ is uniformly almost periodic.

Conversely, suppose $(S(A),G)$ is uniformly almost periodic. Let $A_{\mathbb{R}}$ be the self-adjoint part of A. By Theorem 2.2 it suffices to show that for each $a \in A_{\mathbb{R}}$, the orbit $G[a]$ has compact closure (in $A_{\mathbb{R}}$ or in A). For such an a, let $\hat{a}$ be the map $p \to p(a)$ of $S(A)$ into the real numbers. Then $a \to \hat{a}$ is an isometric linear map of $A_{\mathbb{R}}$ into $C(S(A))$. Thus it suffices to show that for each $a \in A_{\mathbb{R}}$, $\{\widehat{\alpha(a)}\ ;\ \alpha \in G\}$ has compact closure in $C(S(A))$. Since $\widehat{\alpha(a)}(p) = p(\alpha(a)) = \hat{a}(p\alpha)$, we need only show that each $\hat{a}$ is an almost periodic function, which follows from uniform almost periodicity of $(S(A),G)$.

Now suppose that 1) and 2) are satisfied. Since i is continuous into the topology $\mathcal{T}$, it is also continuous into the pointwise topology. Thus $i(\overline{G})$ is pointwise compact. Since $S(A)^{S(A)}$ is pointwise Hausdorff, i is a homeomorphism of $\overline{G}$ onto $i(\overline{G})$. But then $i(G)$ is pointwise dense in both $i(\overline{G})$ and E, so $i(\overline{G}) = E$. Now E is a group, and i is an isomorphism of the semigroup $\overline{G}$ onto E which takes the identity of $\overline{G}$ to the identity of E. If $\alpha \in \overline{G}$, then there exists $\beta \in \overline{G}$ such that $i(\beta) = i(\alpha)^{-1}$. But then β will be an inverse for α, so $\overline{G}$ is a group.

<u>Corollary</u> 2.6. If $\overline{G}$ is compact, then $\overline{G}$ is a subgroup of $\mathrm{Aut}(A)$. In particular, $\overline{G}$ is a compact topological group.

<u>Corollary</u> 2.7. The closure of G in A^A is compact iff the closure of G in $\mathrm{Aut}(A)$ is compact, and in this case the two closures coincide.

<u>Corollary</u> 2.8. If $\overline{G}$ is compact, then every element of E maps the set of pure states of A into itself.

<u>Proof</u>: As $(S(A),G)$ is uniformly almost periodic, the elements of E are invertible maps. By Lemma 2.4, they are affine. Thus each $\alpha \in E$ must take extreme points to extreme points.

<u>Remark</u>: The methods of this section can also be used to obtain analogous results for groups of C^*-automorphisms as defined in [11].

3. Uniformly Almost Periodic C^*-algebras.

In this section we use uniform almost periodicity of $(S(A),G)$ to obtain information about the traces and the ideal structure of the algebra A. We remark that our discussion of centrality is based on that in [13], in which Mosak obtained most of the results of this section for certain group C^*-algebras.

We continue to assume that A is a C^*-algebra with identity I. Moreover, we assume that G is a group of automorphisms of A such that $(S(A),G)$ is uniformly almost periodic. Let μ be normalized Haar measure on $\bar{G}$, and let $a \in A$. As $\alpha \to \alpha(a)$ is continuous on $\bar{G}$, it is weakly μ-measurable. The image of $\bar{G}$ is a compact metric space, hence is separable, so the Bochner integral $\int_{\bar{G}} \alpha(a)\, d\mu(\alpha)$ exists [20, pp. 131-133]. We may thus define a mapping $\#$ of A into A by

$$a^{\#} = \int_{\bar{G}} \alpha(a)\, d\mu(\alpha)\ , \qquad a \in A\ .$$

<u>Lemma</u> 3.1. The mapping $\#$ is a positive, linear, idempotent mapping of A onto $Z_G A$. It is norm-decreasing and takes no non-zero element of A to zero.

<u>Proof:</u> The first statement is proved in [18, Example 1.1]. That $\#$ is norm-decreasing follows from $\|a^{\#}\| \le \int_{\bar{G}} \|\alpha(a)\|\, d\mu(\alpha)$. If $(a^*a)^{\#} = 0$, then for every $p \in S(A)$ we have $\int_{\bar{G}} p(\alpha(a^*a))\,d\mu(\alpha) = 0$. As $\alpha \to p(\alpha(a^*a))$ is positive and continuous, it follows that $(a^*a)^{\#} = 0$ iff $p(\alpha(a^*a)) = 0$ for all $p \in S(A)$ and all $\alpha \in \bar{G}$. Thus $(a^*a)^{\#} = 0$ implies $a^*a = 0$. (See [13, 3.6].)

<u>Lemma</u> 3.2. The mapping r is an affine homeomorphism of $S_G(A)$ onto $S(Z_G A)$.

Proof: r is the inverse of the mapping Φ^* in [18, Example 1.1].

If we wish to study the ideals or traces of A , it is natural to consider the group $\mathcal{G}(A)$ of all inner automorphisms of A . This group is generally too large to act uniformly almost periodically on S(A) . For suppose A is a UHF-algebra (not finite dimensional), and let p be a pure state of A . Then the set of all states of the form $b \to p(ubu^*)$, u unitary in A , is weak*-dense in S(A) . If the action of $\mathcal{G}(A)$ were uniformly almost periodic, then S(A) would be a minimal set [6, 2.5]. But this contradicts the existence of a trace on A . (I am indebted to Erling Størmer for pointing out this counterexample.)

Definition 3.3. Let A be a C^*-algebra with identity. We say that A is uniformly almost periodic if

1) every state of ZA is the restriction of some trace of A , and

2) there exists a group G of inner automorphisms of A such that (S(A),G) is uniformly almost periodic and $Z_G A = ZA$.

Remark 3.4. Let $\mathcal{U}_o$ be a group of unitary elements of A such that the linear span of $\mathcal{U}_o$ is dense in A , and let G_o be the group of all inner automorphisms of A induced by the elements of $\mathcal{U}_o$. Suppose $(S(A),G_o)$ is uniformly almost periodic. Then A is uniformly almost periodic. For if $a \in A$ commutes with every $u \in \mathcal{U}_o$, then $a \in ZA$, so $Z_{G_o} A = ZA$. By Lemma 3.2, restriction takes $S_{G_o}(A)$ onto S(ZA) . If τ is a G_o-invariant state, then for every $a \in A$ and every $u \in \mathcal{U}_o$, we have $\tau(ua - au) = 0$, whence $\tau(ab) = \tau(ba)$ for all $a,b \in A$. Thus

$S_{G_0}(A) = T(A)$. It follows that A is uniformly almost periodic.

We give examples of uniformly almost periodic C^*-algebras in the last section. We assume for the remainder of this section that A is uniformly almost periodic and that G is a group of inner automorphisms of A which satisfies condition 2) of Definition 3.3.

Theorem 3.5. The sets $S_G(A)$ and $T(A)$ coincide, and $ET(A)$ is a weak* closed subset of $T(A)$. (That is, $T(A)$ is a Bauer simplex.) Moreover, $\tau \in T(A)$ is extremal iff $\tau|_{ZA}$ is a character, and r restricted to $ET(A)$ is a homeomorphism onto $ES(ZA)$.

Proof: Since G consists of inner automorphisms, $T(A) \subseteq S_G(A)$. As each $\phi \in S(ZA)$ is the restriction of a trace of A, $r(T(A)) = S(ZA)$. Then $T(A) = S_G(A)$, since r is one-to-one. By Remark 1.5, the restriction of an extremal trace to ZA is a character. If $\tau \in T(A)$ and $r(\tau)$ is pure in $S(ZA)$, then τ must be extremal, since r is affine and one-to-one. Thus r restricts to a bijection of $ET(A)$ and $ES(ZA)$. It follows from Lemma 3.2 that this bijection is a homeomorphism and that $ET(A) = r^{-1}(ES(ZA))$ is weak* closed in $T(A)$.

Lemma 3.6. If $\tau \in ET(A)$, then the left kernel of τ is a primitive ideal of A .

Proof: (after Mosak) Let $\tau|_{ZA} = \phi$. Then ϕ is an irreducible representation of ZA , and we can find an irreducible representation π of A on some Hilbert space $\mathcal{H}_\pi$ such that π is an extension of ϕ .

Then

$$\tau(a^*a) = \tau((a^*a)^\#) = \phi((a^*a)^\#) = \pi((a^*a)^\#) , \quad a \in A .$$

Thus it suffices to show that if π is a representation of A, then $\pi(a) = 0$ iff $\pi((a^*a)^\#) = 0$, or equivalently that $\pi(a^*a) = 0$ iff $\pi((a^*a)^\#) = 0$.

If $x \in \mathcal{H}_\pi$, let $\omega_x(b) = (bx, x)$, $b \in \mathcal{B}(\mathcal{H}_\pi)$. Then

$$\pi((a^*a)^\#) = 0 \iff \omega_x \circ \pi((a^*a)^\#) = 0 \quad \forall x \in \mathcal{H}_\pi \iff$$

$$\int_{\overline{G}} \omega_x \circ \pi(\alpha(a^*a))\, d\mu(\alpha) = 0 \quad \forall x \in \mathcal{H}_\pi \iff$$

$$\omega_x \circ \pi \circ \alpha(a^*a) = 0 \quad \forall\, x \in \mathcal{H}_\pi, \quad \forall \alpha \in \overline{G} \iff \pi \circ \alpha(a^*a) = 0 \quad \forall \alpha \in \overline{G}.$$

Let K be the kernel of π. Since G consists of inner automorphisms and K is a closed ideal, each α in $\overline{G}$ maps K into K. It follows that $\pi \circ \alpha(a^*a) = 0$ $\forall \alpha \in \overline{G}$ iff $\pi(a^*a) = 0$.

Let θ be the mapping of $ET(A)$ into $Prim(A)$ defined by sending an extremal trace into its left kernel, and let ρ be the mapping $P \to P \cap ZA$ of $Prim(A)$ onto $Max(ZA)$. We shall identify a maximal ideal of ZA with the corresponding character. With this identification the mapping r restricted to $ET(A)$ is a homeomorphism of $ET(A)$ and $Max(ZA)$. By Remark 1.5 this homeomorphism factors into $\rho \circ \theta$, i.e. $\tau|_{ZA} = \ker L^\tau \cap ZA$ when $\tau \in ET(A)$. Since r is one-to-one, θ is also one-to-one from $ET(A)$ into $Prim(A)$.

<u>Lemma</u> 3.7. If $\tau \in ET(A)$, then its left kernel is a maximal ideal and θ is a homeomorphism of $ET(A)$ onto $Max(A)$. Moreover, ρ restricts to a homeomorphism of $Max(A)$ onto $Max(ZA)$.

<u>Proof</u>: Let $M \in Max(A)$, and let p be a state of A such that L^p has kernel M. Then $p^\# : a \to p(a^\#)$ is a trace on A. Let $a \in M$. Since the elements of $\overline{G}$ map M into M, p vanishes on each $\alpha(a)$, $\alpha \in \overline{G}$. Thus $p(a^\#) = 0$, and $p^\#$ is a trace

which vanishes on M . The set of all traces which vanish on M is a weak* closed face of $T(A)$. Let τ be any extreme point of this face. Then $\ker L^\tau \supseteq M$, and by maximality of M we have $\ker L^\tau = M$. Thus θ maps $ET(A)$ onto a subspace of $\mathrm{Prim}(A)$ which contains $\mathrm{Max}(A)$.

Let $P \in \mathrm{Prim}(A)$, and suppose there exists $\sigma \in ET(A)$ such that $\ker L^\sigma = P$. Choose $M_P \in \mathrm{Max}(A)$ such that $P \subseteq M_P$, and choose $\tau \in ET(A)$ such that $\ker L^\tau = M_P$. By Remark 1.5 , the characters corresponding to $P \cap ZA$ and $M_P \cap ZA$ are $\sigma|_{ZA}$ and $\tau|_{ZA}$ respectively. Since $M_P \cap ZA = P \cap ZA$, these characters are equal. As r is one-to-one, we have $\sigma = \tau$, hence $M_P = P$. Thus θ is a bijection of $ET(A)$ and $\mathrm{Max}(A)$.

Now ρ is one-to-one on $\mathrm{Max}(A)$, since $r = \rho \circ \theta$, r is one-to-one, and θ is a bijection. As $\mathrm{Max}(A)$ is compact and ρ is continuous, ρ restricts to a homeomorphism ρ_o of $\mathrm{Max}(A)$ onto $\mathrm{Max}(ZA)$. It follows that $\theta = \rho_o^{-1} \circ r$ is also a homeomorphism.

<u>Lemma</u> 3.8. Every primitive ideal of A is maximal. In particular, θ is a homeomorphism of $ET(A)$ and $\mathrm{Prim}(A)$.

<u>Proof</u>: (after Mosak) We define a mapping of $\hat{A}$ into $ET(A)$ as follows. If π is an irreducible representation of A , then $\pi^\#: a \to \pi(a^\#)$ is a trace on A . Suppose $\pi^\# = \frac{1}{2}\tau_1 + \frac{1}{2}\tau_2$ with τ_1 and τ_2 in $T(A)$. Then, as $\pi^\#|_{ZA}$ is a pure state of ZA , $\pi^\# = \tau_1 = \tau_2$ on ZA . Since restriction is one-to-one, we have $\tau_1 = \tau_2 = \pi^\#$ on all of A . If π is unitarily equivalent to π_o, let p and p_o be states associated with π and π_o respectively. Then there exists a unitary u in A such that $p(uau^*) = p_o(a)$ for all $a \in A$. Hence $\pi^\# = p^\# = p_o^\# = \pi_o^\#$, and $\pi \to \pi^\#$ is well defined.

Now $\pi \to \pi^{\#}$ maps onto $ET(A)$. For if $\tau \in ET(A)$, let p be a pure state of A which agrees with τ on ZA . Let $\pi = L^p$, and then $\pi^{\#} = \pi = p = \tau$ on ZA , so $\pi^{\#} = \tau$.

If we can show that the mapping $\pi \to \ker \pi$ of $\hat{A}$ onto $Prim(A)$ is the composition of $\pi \to \pi^{\#}$ and θ , then θ must map onto $Prim(A)$, and hence $Max(A) = Prim(A)$. So we must show that if $\pi \in \hat{A}$, then the kernel of π is $\{a : \pi((a^*a)^{\#}) = 0\}$. But we verified this in the proof of Lemma 3.6.

Theorem 3.9. If A is uniformly almost periodic, then A is a central C^*-algebra.

Proof: Combine Lemmas 3.7 and 3.8.

4. Uniquely Ergodic C^*-Algebras.

We turn now to uniquely ergodic systems and approximately finite C^*-algebras. If X is a compact metric space and T is a homeomorphism of X onto X, then by [15, 2.1] there exists at least one normalized T-invariant Borel measure on X. The system (X,T) is said to be uniquely ergodic if there exists exactly one such measure, or equivalently if $C(X)$ has exactly one T-invariant state. By analogy we define a C^*-algebra to be <u>uniquely ergodic</u> if it possesses exactly one trace.

A C^*-algebra A with identity I is said to be <u>approximately finite</u> if there exists an increasing sequence $\{A_n\}$ of finite dimensional C^*-subalgebras of A, each A_n containing I, such that $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ [3]. We shall see below that every approximately finite C^*-algebra possesses at least one trace, and we shall characterize those which are uniquely ergodic. We assume in this section that A is approximately finite with $\{A_n\}$ and I as above. We note that A is separable, and hence that $S(A)$ is metrizable.

For each $n \geq 1$, the unitary group $\mathcal{U}_n$ of A_n is compact, so there exists a map φ_n of A into A given by

$$\varphi_n : a \to \int_{\mathcal{U}_n} uau^* \, d\mu_n(u) , \qquad a \in A ,$$

where μ_n is normalized Haar measure on $\mathcal{U}_n$.

<u>Lemma</u> 4.1. For each $n \geq 1$, φ_n is a norm-decreasing, idempotent, positive, linear map of A onto $A_n' = \{a \in A : ab = ba$ for all $b \in A_n\}$. Each A_n' is a C^*-subalgebra of A and $\bigcap_{n=1}^{\infty} A_n' = ZA$. If $a,b \in A_n$, then $\varphi_n(ab) = \varphi_n(ba)$.

Proof: If $a \in \bigcap_{n=1}^{\infty} A_n'$, then a commutes with every element of $\bigcup_{n=1}^{\infty} A_n$, hence with every element of A. It is trivial that φ_n is norm-decreasing, and the rest of the lemma follows from [18, Example 1.1].

Lemma 4.2. Let $\{p_n\}$ be a sequence of states of A. Then $\{p_n \circ \varphi_n\}$ is a sequence of states of A and has at least one limit point in $S(A)$. Every limit point is a trace of A.

Proof: Clearly $\{p_n \circ \varphi_n\}$ is a sequence of states, and it has a limit point by compactness of $S(A)$. Let $p_{n_i} \circ \varphi_{n_i} \to \tau$ in $S(A)$. If $a, b \in \bigcup_{n=1}^{\infty} A_n$, then for all sufficiently large n we have $\varphi_n(ab - ba) = 0$, whence $\tau(ab - ba) = 0$. The map $(a,b) \to ab - ba$ is continuous on $A \times A$, so $\tau(ab - ba) = 0$ for all $a, b \in A$.

Corollary 4.3. If ψ is a state of ZA, then there exists a trace τ of A whose restriction to ZA is ψ. If ψ is a character, then τ can be chosen to be extremal.

Proof: Let p be a state of A which extends ψ, and let τ be a weak* limit point of $\{p \circ \varphi_n\}$. Then $\tau = \psi$ on ZA. Suppose now ψ is a character. The set $F = \{\tau \in T(A) : \tau|_{ZA} = \psi\}$ is a non-empty closed face of $T(A)$. Any extreme point of F is an extremal trace which extends ψ.

Remark: The following proposition describes the approximately finite C^*-algebras which possess a centering map analogous to the map $^\#$ of the last section. We note however that the map $a \to \varphi(a)$ below may annihilate some non-zero positive elements of A.

Proposition 4.4. The following are equivalent:

1) For each $a \in A$ the sequence $\{\varphi_n(a)\}$ converges in norm to an element $\varphi(a)$ of A.

2) The mapping $r: \tau \to \tau|_{ZA}$ of $T(A)$ onto $S(ZA)$ is one-to-one.

If these conditions are satisfied, then for each $a \in A$ we have $\varphi(a) \in ZA$, and for each $p \in S(A)$ the mapping $a \to p(\varphi(a))$ is a trace.

Proof: 1) => 2): Let $a \in A$. If $\sigma \in T(A)$, then for each n we have $\sigma(\varphi_n(a)) = \sigma(a)$, so $\sigma(a) = \sigma(\varphi(a))$. Suppose σ and τ are in $T(A)$ and $\sigma = \tau$ on ZA. As $\varphi(a) \in \bigcap_{n=1}^{\infty} A_n' = ZA$, $\sigma(a) = \tau(a)$. Thus r is one-to-one. That $a \to p(\varphi(a))$ is a trace follows from the fact that φ is positive and linear and vanishes on $ab - ba$ for all $a, b \in \bigcup_{n=1}^{\infty} A_n$.

2) => 1): If r is one-to-one, then it is an affine homeomorphism of $T(A)$ onto $S(ZA)$, and its restriction to $ET(A)$ is a homeomorphism onto $ES(ZA)$. We use this homeomorphism and the Gelfand transform $\wedge$ to identify ZA with $C(ET(A))$. For each $a \in A$, put $a^{\#}(\tau) = \tau(a)$, $\tau \in ET(A)$. Then $a^{\#} \in ZA$, and the mapping $a \to a^{\#}$ is linear, norm-decreasing, positive, and invariant under the inner automorphisms of A. As $z^{\#}(\tau) = \hat{z}(r(\tau))$, $z^{\#} = z$ for $z \in ZA$. To show that $\varphi_n(a)$ is convergent for each $a \in A$, it suffices to show that for each positive $a \in A$, $\|\varphi_n(a) - a^{\#}\| \to 0$, i.e. $\sup_{p \in S(A)} |p \circ \varphi_n(a) - p(a^{\#})| \to 0$. If this is false, then there exist $a_0 \geq 0$ in A, a subsequence $\{\varphi_{n_i}\}$ of $\{\varphi_n\}$, and $p_i \in S(A)$ such that

$$|p_i \circ \varphi_{n_i}(a_0) - p_i(a_0^{\#})| \geq \varepsilon > 0 \quad \text{for all } i \geq 1 \tag{1}$$

By passing to a subsequence we may assume $p_i \circ \varphi_{n_i} \to \tau$ and $p_i \to p_o$ in $S(A)$. Then τ and $p_o^{\#}: a \to p_o(a^{\#})$ are traces of A. For $z \in ZA$ we have

$$\tau(z) = \lim_{i\to\infty} p_i \circ \varphi_{n_i}(z) = \lim_{i\to\infty} p_i(z) = p_o(z) = p_o(z^{\#}) ,$$

so $r(\tau) = r(p_o^{\#})$. But then $\tau = p_o^{\#}$, which contradicts (1).

If $a \in A$, let $\hat{a}$ be the mapping $p \to p(a)$ of $S(A)$ into $\mathbb{C}$. The following is a C^*-algebraic analogue of [15, 5.3].

Theorem 4.5. If A is an approximately finite C^*-algebra, then the following are equivalent:

1) A is uniquely ergodic;

2) For each $a \in A$ the sequence $\{\widehat{\varphi_n(a)}\}$ converges uniformly on $S(A)$ to a constant function;

3) For each $a \in A$ there exists a subsequence of $\{\widehat{\varphi_n(a)}\}$ which converges pointwise on $S(A)$ to a constant function.

If these conditions are satisfied, then the constant function of conditions 2) and 3) has the value $\tau(a)$, where τ is the trace of A.

Proof: 1) => 2): Let $a \in A$. By Corollary 4.3, $ZA = \mathbb{C}I$, and by the last proposition $\varphi_n(a)$ converges in norm to $K_a I$ for some complex number K_a. Thus $\sup_{p\in S(A)} |p\circ\varphi_n(a) - K_a| \to 0$.

2) => 3): Trivial

3) => 1): Let $a \in A$, $\sigma \in T(A)$, and suppose $\{\widehat{\varphi_{n_i}(a)}\}$ converges pointwise to the constant function K_a. Then $\sigma(a) = \sigma(\varphi_{n_i}(a)) \to K_a$, so $a \to K_a$ is the only trace on A.

5. Examples.

Example 5.1. Let X be a compact Hausdorff space and $A = C(X)$. Let (X,Γ) be a transformation group. As in the introduction we let tf be the function

$$(tf)(x) = f(xt) \qquad x \in X,$$

where $t \in \Gamma$ and $f \in C(X)$. Let G be the group of all automorphisms of A which have the form $f \to tf$, $t \in \Gamma$. It follows from Remark 1.4 and Theorems 2.2 and 2.5 that (X,Γ) is uniformly almost periodic iff $(S(A),G)$ is uniformly almost periodic. It is not difficult to show that if these two transformation groups are uniformly almost periodic, then their enveloping semigroups are homeomorphic and isomorphic.

Example 5.2. Let A be a UHF-algebra. We may write $A = \overline{\bigcup_{n=1}^{\infty} A_n}$, where $A_n = M_1 \otimes \ldots \otimes M_n$ and for each $i \geq 1$, M_i is a finite dimensional factor. Let $\mathcal{U}$ be the group of all unitaries in $\bigcup_{n=1}^{\infty} A_n$ which have the form $u_1 \otimes \ldots \otimes u_k$, where $k \geq 1$ and u_i is a unitary element of M_i, $i = 1,2,\ldots,k$. If G is the group of all inner automorphisms of A induced by elements of $\mathcal{U}$, then we claim that $(S(A),G)$ is uniformly almost periodic. By Theorems 2.2 and 2.5, it suffices to show that if $a \in \bigcup_{n=1}^{\infty} A_n$, then the set $\{uau^* : u \in \mathcal{U}\}$ has compact closure in A. Now $\mathcal{U}$ leaves the generating set $\{a_1 \otimes \ldots \otimes a_n : a_i \in M_i,\ i = 1,\ldots,n\}$ of A_n invariant, hence leaves A_n invariant, $n \geq 1$. It follows that $\{uau^* : u \in \mathcal{U}\}$ lies in the closed ball of radius $\|a\|$ in some A_n, hence has compact closure, since A_n is finite dimensional.

Example 5.3. Let Γ be a discrete group, and let A be the group C^*-algebra of Γ [5, 13.9.1]. We identify $L^1(\Gamma)$

with a dense $*$-subalgebra of A, and for each $g \in \Gamma$ we write δ_g for the function which is one at g and zero elsewhere on Γ. Then Γ is isomorphic to the subgroup $\{\delta_g : g \in \Gamma\}$ of the unitary group of A, and we also identify these groups. Then Γ has dense linear span in A.

Let $\mathrm{Aut}(\Gamma)$ be the group of all automorphisms of Γ. Each $\alpha \in \mathrm{Aut}(\Gamma)$ has a unique extension to an automorphism $\tilde{\alpha}$ of A, and $\alpha \to \tilde{\alpha}$ is a one-to-one group homomorphism from $\mathrm{Aut}(\Gamma)$ into $\mathrm{Aut}(A)$.

A group is said to be <u>class-finite</u> if every conjugacy class in the group is a finite set, i.e. every element has a finite orbit under the action of the inner automorphisms. Let G be the group of all inner automorphisms of Γ. By Theorems 2.2 and 2.5, $(S(A),\tilde{G})$ is uniformly almost periodic iff the orbit of each δ_g has compact closure. Since Γ is a discrete subset of A, $(S(A),\tilde{G})$ is uniformly almost periodic iff Γ is class-finite.

We remark that class-finite groups are precisely the discrete $[FIA]^-$-groups studied by Mosak in [13]. Thoma studied harmonic analysis on class-finite groups in [19], and Neumann gave a structure theory for such groups in [14].

<u>Remark</u> 5.4. The algebras given in these three examples are uniformly almost periodic C^*-algebras: In 5.1 put $\mathcal{U}_0$ equal to the unitary group of A, in 5.2 put $\mathcal{U}_0 = \mathcal{U}$, and in 5.3 put $\mathcal{U}_0 = \{\delta_g : g \in \Gamma\}$. In all three cases $\mathcal{U}_0$ has dense linear span and we may apply Remark 3.4.

References

1. J.F. Aarnes, E.G. Effros and O.A. Nielsen, Locally compact spaces and two classes of C^*-algebras, Pacific J. Math. 34 (1970), 1-16.

2. E.M. Alfsen, "Compact Convex Sets and Boundary Integrals", Springer-Verlag, Berlin, 1972.

3. O. Bratteli, Inductive limits of finite dimensional C^*-algebras, Trans. Amer. Math. Soc. 171 (1972), 195-234.

4. J. Dixmier, "Les algèbres d'opérateurs dans l'espace Hilbertien (algèbres de von Neumann)", Gauthier-Villars, Paris, 2nd ed., 1969.

5. J. Dixmier, "Les C^*-algèbres et leurs représentations", Gauthier-Villars, Paris, 2nd. ed. 1969.

6. R. Ellis, "Lectures on Topological Dynamics", W.A. Benjamin, New York, N.Y., 1969.

7. E.G. Effros and F. Hahn, Locally compact transformation groups and C^*-algebras, Mem. Amer. Math. Soc. 75 (1967).

8. J.G. Glimm, On a certain class of operator algebras, Trans. Amer. Math. Soc. 95 (1960), 318-340.

9. S. Grosser and M. Moskowitz, On central topological groups, Trans. Amer. Math. Soc. 127 (1967), 317-340.

10. R.V. Kadison, The trace in finite operator algebras, Proc. Amer. Math. Soc. 12 (1961), 973-977.

11. R.V. Kadison, Transformations of states in operator theory and dynamics, Topology. 3, Supplement 2 (1965), 177-198.

12. J.L. Kelley, "General Topology", D. van Nostrand Co., Princeton, N.J., 1955.

13. R.D. Mosak, The L^1- and C^*-algebras of $[FIA]_B^-$-groups and their representations, Trans. Amer. Math. Soc. 163 (1972), 277-310.

14. B.H. Neumann, Groups with finite classes of conjugate elements, Proc. London Math. Soc. (3) 1 (1951), 178-187.

15. J.C. Oxtoby, Ergodic sets, Bull. Amer. Math. Soc. 58 (1952), 116-136.

16. L.S. Pontryagin, "Topological Groups", Gordon and Breach, New York, N.Y., 2nd. ed., 1966.

17. E. Størmer, Invariant states of von Neumann algebras, Math. Scand. 30 (1972), 253-256.

18. E. Størmer, Large groups of automorphisms of C^*-algebras. Comm. Math. Phys. 5 (1967), 1-22.

19. E. Thoma, Zur harmonischen Analyse klassenfiniter Gruppen, Invent. Math. 3 (1967), 20-42.

20. K. Yosida, "Functional Analysis", Springer-Verlag, Berlin, 1965.